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Heat Production in a Simply Supported Multilayer Elliptic Annulus Composite Plate and Its Associated Thermal Stresses

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Abstract

This paper is concerned with the theoretical treatment of thermoelastic problem in multilayer elliptical composite plate and the generation of heat in the body as well as at the interfaces with imperfect thermal contact under arbitrary initial temperature distribution. In order to obtain a closed-form solution of transient heat conduction problem, an alternative approach using a new Sturm-Liouville integral transform is presented that considers the series expansion using the eigenfunction expansion method for Sturm-Liouville boundary value problem. Any particular case of special interest can be derived from assigning suitable values to the parameters and functions of the temperature field and its associated stresses. As a particular case, the quandary on the heat conduction and its stresses on a two-layered elliptical plate were solved.

Nomenclature

ξ, η	Elliptical Coordinates	q	Parameter of Mathieu equation
$ce_n(\eta, q)$	Ordinary Mathieu function of first kind of order n	$Ce_n(\xi, q)$	Modified Mathieu function of second kind of order n
h	Interfocal distance (and $1/h^2 = c^2(\cosh\xi - \cos 2\eta)/2$)	$q_{n,m}$	The temperature distribution at any time t
$\bar{f}(q_n, m)$	Mathieu transform of $f(\xi, eta)$	$\theta(\xi, \eta, t)$	Parametric roots of equation (A7)
ϕ	The Airy's stress functions	T_0	The reference temperature
$2c$	Focal length$= 2\sqrt{a_i^2 - b_i^2} = 2\sqrt{a_o^2 - b_o^2}$	$\varepsilon_i, \varepsilon_0$	$=\tan^{-1}(b_i/a_i) = \tanh^{-1}(b_o/a_o)$

1. Introduction

The utilization of composite materials in aeronautic industries, submarines, automotive engineering, sport equipments etc. has noticeably progressed. Therefore, composites can be considered as one of the most widely used materials because of their adaptability to different situations and the relative ease of combination with other materials to serve specific purposes and exhibit desirable properties. This remarkable usage of such kind of materials is due to its high strength and having the high module with low density. Hence in

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many applications, utilization of composite materials as preferred compared to isotropic materials. Therefore, a number of theoretical studies concerning them have been reported so far. For example, Ölçer [1] presented an analytical study for the distribution of three-dimensional unsteady temperatures in a hollow right circular cylinder of finite length and composed of two concentric radial layers in imperfect thermal contact. Wankhede and Bhonsale [2] discussed the solution of heat conduction on multilayered composite plates, cylinders, or spheres consisting of k-layers utilizing an incipient integral transform which is more analogues compared to the classical method of Sturm-Liouville [3] system. Malzbender [4] obtained the general solution for elastic deformation of monolithic and multilayered materials due to external loads and moments, a mismatch between thermal expansion and temperature gradients. Kalamkarov et al. [5] developed a new method for solving steady-state heat conduction for multilayer composite wedge-shaped bodies based on a generalization method of the integral Mellin transform. Lu et al. [6] provided a novel analytical method for the quandary of transient heat conduction in a one-dimensional hollow composite cylinder with a time-dependent boundary temperature. Norouzi et al. [7] obtained an exact analytical solution for steady conductive heat transfer in multilayer spherical fibre reinforced composite laminates by utilizing the separation of variables method and the set of equations cognate to the coefficient of Fourier-Legendre series. Singh [8] discussed the finite integral transform method to solve the two-dimensional multilayer heat conduction problem in spherical and cylindrical coordinates with time-dependent boundary conditions and/or heat sources. In this paper, the eigenfunction expansion approach satisfying periodic boundary condition in the angular direction has been applied. Kayhani et al. [9] presented a steady analytical solution for heat conduction in a cylindrical multilayer composite laminate in which the fibre direction varied between layers. The analytical solution of the governing equation was obtained utilizing Strum-Liouville theorem to derive an felicitous Fourier transformation. Dalir and Nourazar [10] presented an exact analytical solution using the eigenfunction expansion method for the problem on three-dimensional transient heat conduction in a cylinder with multiple radial layers in which time-dependent, spatially non-uniform internal volume heat sources were installed. Wange and Gaikwad [11] obtained an analytical solution for non-homogeneous, one-dimensional, transient heat conduction problem in the composite region based on the method of separation of variables and of orthogonal expansion of functions over multilayer regions. Assouane et al. [12] investigated a transient semi-analytical solution for heat conduction with general linear boundary conditions in a cylindrical multilayered composite layer. Hitherto, a plethora of researches has been carried out on the mechanical and thermomechanical behaviour of multilayered composites of the different geometrical profile, while very few works are available on the elliptical structure. It may be due to the mathematical complications; closed-form solutions for heat conduction problems in an elliptical composite object are recherche in literature. These elliptical plates and cylinders of composite materials due to their elementary geometries are widely utilized as structural elements in various applications. During literature review, only a few reports were observed which extensively studied the multilayer composite with elliptical objects. For example, Mansfield [13] analyzed the case of the linearly varying load applied to a clamped elliptical multi-layered plate exhibiting general coupling between moments and planar strains. Vodika [14] used the classical method to determine the steady temperature distribution in a finite elliptic cylinder consisting of any number of plane-parallel layers in consideration with adjacent tight layer contacts. Vasilenko and Urusova [15] considered the solution for the freely supported multilayered elliptical plate in a stressed state with a rigid contour fixation. Vasilenko [16] proposed a solution for determining the temperature fields and stresses in orthotropic elliptic plates whose principal axes of elasticity and thermal conduction do not coincide with the axis of the ellipse. Most of the studies considered by authors above have not considered any thermoelastic problem for multilayer elliptical annulus composite plate subjected to the generation of heat in the body as well as at the interfaces with imperfect thermal contact under arbitrary initial temperature distribution.

This paper proposes a new analytical method to determine more general closed-form solutions by establishing Sturm-Liouville integral transform. The consequentiality of proposed transform over the previously published techniques [2, 14] can be seen while obtaining the temperature of any height for composite object profile, defined in elliptical coordinates (ξ, η, z) by $\xi_i < \xi < \xi_{i+1}$, $0 < \eta < 2\pi$, $0 < z < \ell$. Furthermore, by considering small deflection theory, it is recommended study the thermally induced deflection of a plate with elastic supports at both boundaries using the theory of integral transform. Moreover, the intensities of bending moments, twisting moments, shearing forces and effective shear forces were formulated involving the ordinary Mathieu and modified Mathieu functions and their derivatives. The analytical solution for the thermal stress components was obtained in terms of resultant forces and resultant moments. The prosperity of this novel research mainly lies in the incipient mathematical procedures which present a much simpler approach for optimisation of the design regarding material utilization and performance in engineering quandary, categorically for the tenacity of the thermoelastic deportment in elliptical plate engaged as the substructure of pressure vessels, furnaces and so forth.

Formulation of the Problem

The geometry of the plate indicates that an elliptical coordinate system is the most appropriate choice for the reference frame, which is related to the rectilinear coordinate (x, y, z) by the relation

$$x = \cosh\xi, \quad y = c\sinh\xi\sin\eta, \quad z = z, \quad c = (a^2 - b^2)^{1/2} \tag{1}$$

in which c is the semi-focal length of the ellipse, a and b are semi-major and semi-minor axis respectively. An elliptical composite plate was considered which occupies space $D : \{(\xi, \eta, z) \in R^3 : \ \xi_i < \xi < \xi_{i+1}, 0 < \eta < 2\pi, 0 < z < \ell\}, 1 \le i \le n$ as shown in Fig. 1. It was assumed that each rigid layer in each composite body is homogeneous and isotropic, with thermal properties independent of temperature and that the layers are in imperfect thermal contact at the interfaces, which is characterized by the finite interfacial conductances $h_i > 0, i = 1, 2, \cdots, (k-1)$.

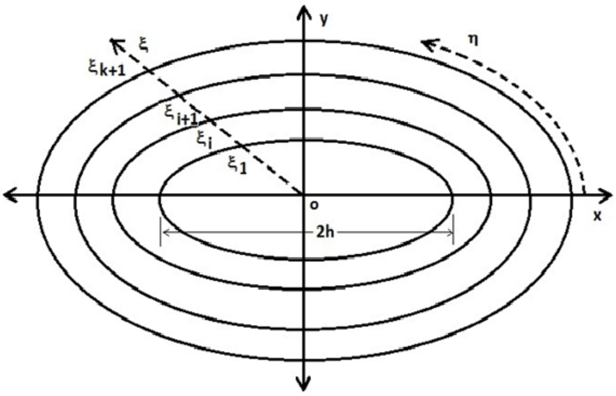

Fig. 1. Elliptical composite configuration profile.

The differential equation governing transient temperature distribution, $\theta_i(\xi, \eta, z, t)$, with an internal heat source for a thin elliptical annulus composite plate in the i^{th} layer can be defined as

$$\nabla^2\theta_i + \frac{\partial^2\theta_i}{\partial z^2} + \frac{Q_i(\xi,\eta)\delta(z-z_0)\delta(t)}{\lambda_i} = \frac{1}{\kappa_i}\frac{\partial\theta_i}{\partial t} \tag{2}$$

subjected to initial and boundary conditions

$$\theta_i(\xi, \eta, z, 0) = \theta_o \tag{3}$$

$$\left.\begin{aligned}
&h_0\theta_i(\xi,\eta,z,t) - \alpha_1\theta_i(\xi,\eta,z,t)_{,\xi}\big|_{\xi=\xi_1} \\
&= f_1(z,t), \quad h_0 \ge 0, \\
&h_k\theta_k(\xi,\eta,z,t) + \alpha_k\theta_k(\xi,\eta,z,t)_{,\xi}\big|_{\xi=\xi_{k+1}} \\
&= f_2(z,t), \quad h_k \ge 0, \\
&\alpha_i\theta_i(\xi,\eta,z,t)_{,\xi}\big|_{\xi=\xi_{k+1}} \\
&= \alpha_{i+1}\theta_{i+1}(\xi,\eta,z,t), \ _{\xi}\big|_{\xi=\xi_{i+1}} \\
&= [\theta_{i+1}(\xi,\eta,z,t) - \theta(\xi,\eta,z,t)]/R_i\big|_{\xi=\xi_{i+1}}, \\
&i = 1, 2, \cdots, (k-1)
\end{aligned}\right\} \tag{4}$$

$$\theta_i(\xi,\eta,z,t)_{,z}\big|_{z=0} = 0, \ \theta_i(\xi,\eta,z,t)\big|_{z=\ell} = 0 \tag{5}$$

where the prime (,) in equations denotes differentiation with respect to the variable specified in the subscript; the Laplacian operator in elliptical coordinates is represented as

$$\nabla^2 = h^2(\partial_{,\xi\xi} + \partial_{,\eta\eta}) \tag{6}$$

where $f_1(z,t)$ and $f_2(z,t)$ represent sectional heat supply, $Q_i(\xi,\eta)\delta(z-z_0)\delta(t)$ denotes distributed heat source, θ_0 is the surrounding temperature, $\delta(\ )$ is the Dirac delta function in which $\xi \ne \xi_0, \ \xi_0 \in [\xi_i, \xi_{i+1}]$, $\eta \ne \eta_0, \ \eta_0 \in [0, 2\pi]$ and $z \ne z_0, \ z_0 \in [0, \ell]$, λ_i for thermal conductivity and heat capacity per unit volume $(\rho C)_i$ with ρ_i for density and C_i as specific heat for the i^{th} layer, respectively. The physical significance of the interface boundary conditions [i.e. equation (4)] is as follows - (i) the finite value of the layer coefficient $h_i > 0, \ i = 1, 2, \cdots (k-1)$ in the first two lines of equation (4) represents a discontinuity of temperature at the corresponding interface, and (ii) last line of equation (4) implies that the heat flux is continuous at the same interface or perfect thermal conduct there.

According to the classical small deflection theory [18], the differential equation of motion of a heated elliptic plate can be written as

$$D_i\nabla^4 w_i(\xi,\eta,t) = -\frac{1}{1-\upsilon}\nabla^2 M\theta_i \tag{7}$$

where D is the flexural stiffness of the plate given as $D_i = E_i\ell^3/12(1-\upsilon^2)$, υ denotes the Poisson's ratio and resultant moment and resultant force can be denoted as

$$\left.\begin{aligned}
M_\theta^{(i)} &= \alpha_i E_i \int_0^\ell z\theta_i dz \\
N_\theta^{(i)} &= \alpha_i E_i \int_0^\ell \theta_i dz
\end{aligned}\right\} \tag{8}$$

with α_i and E_i denoting coefficient of linear thermal expansion and Young's modulus of the material of the plate respectively. Case of the study was restricted to the symmetrical deflection due to the complexity of equations, that is to say, the transversal displacement depends only on the radius and time.
The initial boundary conditions are given as

$$w_i(\xi,\eta,t)\big|_{t=0} = w_i(\xi,\eta,t)_{,t}\big|_{t=0} = 0 \tag{9}$$

Then, it was assumed that the movement of boundaries, $\xi = \xi_1$ and $\xi = \xi_{k+1}$, is limited by an elastic reaction under thermal load. Therefore shall satisfy the conditions of continuity and the boundary condition are as follows

$$\left.\begin{aligned}
&w_i(\xi,\eta,t) + w_i(\xi,\eta,t)_{,\xi}\big|_{\xi=\xi_1} = 0, \\
&w_k(\xi,\eta,t) + w_k(\xi,\eta,t)_{,\xi}\big|_{\xi=\xi_{k+1}} = 0, \\
&w_i(\xi,\eta,t)_{,\xi}\big|_{\xi=\xi_{i+1}} = w_{i+1}(\xi,\eta,t)_{,\xi}\big|_{\xi=\xi_{i+1}} \\
&= h_i[w_{i+1}(\xi,\eta,t) - w_i(\xi,\eta,t)]\big|_{\xi=\xi_{i+1}}, \\
&i = 1, 2, \cdots (k-1)
\end{aligned}\right\} \tag{10}$$

The proportionality constants of the support given by the Hooke's law were assumed to be unity, that is dependent on the elastic properties of the surrounding medium and they, in general, are different.

Now the basic equations of resultant forces, resultant bending moments per unit width, transverse shearing forces and the effective shear force intensity are defined as

$$N_\xi^{(i)} = N_\eta^{(i)} = N_{\xi\eta}^{(i)} = 0 \tag{11}$$

$$M_\xi^{(i)} = \frac{-2D_i}{c^2(\cosh 2\xi - \cos 2\eta)}\left\{\left(\frac{\partial^2 w_i}{\partial \xi^2} + v\frac{\partial^2 w_i}{\partial \eta^2}\right) + \frac{(1-v)}{(\cosh 2\xi - \cos 2\eta)}\left(\sin 2\eta\frac{\partial w_i}{\partial \eta} - \sinh 2\xi\frac{\partial w_i}{\partial \xi}\right)\right\} - \frac{M_\theta^{(i)}}{1-v} \tag{12}$$

$$M_\eta^{(i)} = \frac{-2D_i}{c^2(\cosh 2\xi - \cos 2\eta)}\left\{\left(v\frac{\partial^2 w_i}{\partial \xi^2} + \frac{\partial^2 w_i}{\partial \eta^2}\right) + \frac{(1-v)}{(\cosh 2\xi - \cos 2\eta)}\left(\sinh 2\xi\frac{\partial w_i}{\partial \xi} - \sin 2\eta\frac{\partial w_i}{\partial \eta}\right)\right\} - \frac{M_\theta^{(i)}}{1-v} \tag{13}$$

$$M_{\xi\eta}^{(i)} = -\frac{2D_i(1-v)}{c^2(\cosh 2\xi - \cos 2\eta)}\left\{\frac{\partial w_i}{\partial \xi}\sin 2\eta + \frac{\partial w_i}{\partial \eta}\sinh 2\xi - \frac{\partial^2 w_i}{\partial \xi \partial \eta}(\cosh 2\xi - \cos 2\eta)\right\} \tag{14}$$

$$Q_\xi^{(i)} = \frac{2\sqrt{2}D_i}{c^3|\cosh 2\xi - \cos 2\eta|^{5/2}}\left\{2\sinh 2\xi\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w_i}{\partial \eta^2}\right) - (\cosh 2\xi - \cos 2\eta)\frac{\partial}{\partial \xi}\left(\frac{\partial^2 w_i}{\partial \xi^2} + \frac{\partial^2 w_i}{\partial \eta^2}\right)\right\} \tag{15}$$

$$Q_\eta^{(i)} = \frac{2\sqrt{2}D_i}{c^3|\cosh 2\xi - \cos 2\eta|^{5/2}}\left\{2\sinh 2\eta\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w_i}{\partial \eta^2}\right) - (\cosh 2\xi - \cos 2\eta)\frac{\partial}{\partial \eta}\left(\frac{\partial^2 w_i}{\partial \xi^2} + \frac{\partial^2 w_i}{\partial \eta^2}\right)\right\} \tag{16}$$

$$V_\xi^{(i)} = Q_\xi^{(i)} - \frac{\sqrt{2}}{c|\cosh 2\xi - \cos 2\eta|^{1/2}}\frac{\partial M_{\xi\eta}^{(i)}}{\partial \eta} \tag{17}$$

The equation (12) must be satisfied for the simply supported plate as

$$M_\xi^{(i)}\big|_{\xi=\xi_1} = 0, \quad \text{for all } \eta \text{ in } 0 \le \eta \le 2\pi \tag{18}$$

Furthermore, the thermal stress components in terms of resultant forces and resultant moments are given as

$$\sigma_{\xi\xi}^{(i)} = \frac{1}{\ell}N_\xi^{(i)} + \frac{12z}{\ell^3}M_\xi^{(i)} + \frac{1}{1-v}\left(\frac{1}{\ell}N_\theta^{(i)} + \frac{12z}{\ell^3}M_\theta^{(i)} - \alpha_i E_i \theta_i\right) \tag{19}$$

$$\sigma_{\eta\eta}^{(i)} = \frac{1}{\ell}N_\eta^{(i)} + \frac{12z}{\ell^3}M_{\eta\eta}^{(i)} + \frac{1}{1-v}\left(\frac{1}{\ell}N_\theta^{(i)} + \frac{12z}{\ell^3}M_\theta^{(i)} - \alpha_i E_i \theta_i\right) \tag{20}$$

$$\sigma_{\xi\eta}^{(i)} = \frac{1}{\ell}N_{\xi\eta}^{(i)} - \frac{12z}{\ell^3}M_{\xi\eta}^{(i)} \tag{21}$$

The equations (1) to (21) constitute the mathematical formulation of the problem under consideration.

Solution to the Problem

In order to solve fundamental differential equation (2) using the theory of integral transformation, firstly, a new integral transform of order n and m was introduced over the variable ξ and η as

$$\bar{f}_i(q_{2n}, m) = \beta_i \int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}} (\cosh 2\xi - \cos 2\eta)\Phi_{i,n,m}(\xi,\eta) f_i(\xi,\eta)\, d\xi d\eta \tag{22}$$

where $\Phi_{i,n,m}(\xi,\eta)$ is the kernel and $\bar{f}_i(q_{2n,m})$ is Sturm-Liouville transform for the composite region (refer Appendix A).

The inversion theorem is given by

$$f_i(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{2n}\Phi_{i,n,m}(\xi,\eta)\sum_{i=1}^{k}\bar{f}_i(q_{2n,m}) \tag{23}$$

Applying the new Sturm-Linville transform defined in equation (22) to equation (2), we obtain

$$\frac{\partial^2 \bar{\theta}_i}{\partial z^2} - \left(\frac{4q_{2n,m}\bar{\theta}_i}{c^2} + G(q_{2n,m}, z, t)\right) + \frac{\delta(z-z_0)\delta(t)\bar{Q}_i(q_{2n,m})}{\lambda_i} = \frac{1}{\kappa_i}\frac{\partial \bar{\theta}_i}{\partial t} \tag{24}$$

in which

$$G(q_{2n,m}, z, t) = 4\pi A_0^{(2n)}\{\psi_{k,n,m}(\xi_{k+1}) f_2(z,t) + \psi_{1,n,m}(\xi_1) f_1(z,t)\}/c^2$$

and

$$\bar{Q}_i(q_{2n,m}) = \beta_i \int_0^{2\pi} \int_{\xi_i}^{\xi_{i+1}} (\cosh 2\xi - \cos 2\eta)\Phi_{i,n,m}(\xi,\eta) Q_i(\xi,\eta) d\xi d\eta$$

Applying the Fourier integral transform stated in [4] in equations (24) and (3), and using equation (5), one leads to

$$\frac{d\bar{\bar{\theta}}_i}{dt} + \alpha_{2n,m}^2 \bar{\bar{\theta}}_i = \sqrt{\frac{2}{\ell}} \left(\frac{\kappa_i \cos(\beta_p z_0)\delta(t)\bar{Q}_i(q_{2n,m})}{\lambda_i} \right) + \bar{G}(q_{2n,m}, \beta_p, t) \tag{25}$$

in which

$$\alpha_{2n,m}^2 = \kappa_i \left(\frac{4q_{2n,m}}{c^2} + \beta_p^2 \right)$$

The differential equation in $\bar{\bar{\theta}}_i(q_{2n,m}, \beta_p, t)$ is transformed by means of Laplace transform and convolution theorem, it can be noticed that

$$\bar{\theta}_i(q_{2n,m}, \beta_p, t) = \exp[-\alpha_{2n,m}^2 t] \left[\theta_0 + \sqrt{\frac{2}{\ell}} \frac{\kappa_i}{\lambda_i} \cos(\beta_p z_0) \bar{Q}_i(q_{2n,m}) - \int_0^\tau \exp[\alpha_{2n,m}^2 \tau] \bar{G}(q_{2n,m}, \beta_p, \tau) d\tau \right] \tag{26}$$

and then accomplishing inversion theorems of the Fourier transform rules on equation (26), we obtain

$$\bar{\theta}_i(q_{2n,m}, z, t) = \exp[-\alpha_{2n,m}^2 t] \sqrt{\frac{2}{\ell}} \sum_{p=1}^{\infty} \cos(\beta_p z) \left[\theta_0 + \sqrt{\frac{2}{\ell}} \frac{\kappa_i}{\lambda_i} \cos(\beta_p z_0) \times \bar{Q}_i(q_{2n,m}) - \int_0^\tau \exp[\alpha_{2n,m}^2 \tau] \bar{G}(q_{2n,m}, \beta_p, \tau) d\tau \right] \tag{27}$$

Finally, through the inversion theorem defined in equation (23), it results in

$$\bar{\theta}_i(\xi,\eta,z,t) = \sqrt{\frac{2}{\ell}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{2n} \Phi_{i,n,m}(\xi,\eta) \sum_{i=1}^{k} \exp[-\alpha_{2n,m}^2 t] \times \sum_{p=1}^{\infty} \cos(\beta_p z) \left[\theta_0 + \sqrt{\frac{2}{\ell}} \frac{\kappa_i}{\lambda_i} \cos(\beta_p z_0) \bar{Q}_i(q_{2n,m} - \int_0^\tau \exp[\alpha_{2n,m}^2 \tau] \bar{G}(q_{2n,m}, \beta_p, \tau) d\tau \right] \tag{28}$$

The above function, which is given in equation (28), represents the temperature at every instance and at all points of the elliptical annulus of finite height under the influence of radiations type boundary conditions.

Substituting equation (28) into equation (8), leads to

$$M_\theta^{(i)} = \sqrt{\frac{2}{\ell}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{i=1}^{k} \exp[-\alpha_{2n,m}^2 t] \alpha_i E_i C_{2n} \Phi_{i,n,m}(\xi,\eta) \times \left\{ \sum_{p=1}^{\infty} \left[(\cos\beta_p \ell + \beta_p \ell \sin\beta_p \ell - 1)/\beta_p^2 \right] \left[\theta_0 + \sqrt{\frac{2}{\ell}} \frac{\kappa_i}{\lambda_i} \cos(\beta_p z_0) \times \bar{Q}_i(q_{2n,m}) \right] - \int_0^\tau \exp[\alpha_{2n,m}^2 \tau] \bar{G}(q_{2n,m}, \beta_p, \tau) d\tau \right\} \tag{29}$$

$$N_\theta^{(i)} = \sqrt{\frac{2}{\ell}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{i=1}^{k} \exp[-\alpha_{2n,m}^2 t] \alpha_i E_i C_{2n} \Phi_{i,n,m}(\xi,\eta) \times \left\{ \sum_{p=1}^{\infty} \frac{\sin\beta_p \ell}{\beta_p} \left[\theta_0 + \sqrt{\frac{2}{\ell}} \frac{\kappa_i}{\lambda_i} \cos(\beta_p z_0) \bar{Q}_i(q_{2n,m}) \right] - \int_0^\tau \exp[\alpha_{2n,m}^2 \tau] \bar{G}(q_{2n,m}, \beta_p, \tau) d\tau \right\} \tag{30}$$

Substituting equation (29) in equation (7), and using equations (9)-(10), yields

$$w_i(\xi,\eta,t) = \sqrt{\frac{2}{\ell}} \frac{c^2 l^3}{48(1+\upsilon)} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{i=1}^{k} \{\exp[-\alpha_{2n,m}^2 t] - 1\} \alpha_i \times C_{2n} \Phi_{i,n,m}(\xi,\eta) \left\{ \sum_{p=1}^{\infty} \left[(\cos\beta_p \ell + \beta_p \ell \sin\beta_p \ell - 1)/\beta_p^2 \right] \times \left[\bar{\bar{\theta}}_0 + \sqrt{\frac{2}{\ell}} \frac{\kappa_i}{\lambda_i} \delta(t) \cos(\beta_p z_0) \bar{Q}_i(q_{2n,m}) \right] - \int_0^\tau \exp[\alpha_{2n,m}^2 \tau] \bar{G}(q_{2n,m}, \beta_p, \tau) d\tau \right\} / q_{2n,m} \tag{31}$$

Consequently, the expressions (12)-(14) become

$$
\begin{aligned}
M_{\xi}^{(i)} &= \sqrt{\frac{2}{\ell}}\frac{1}{1-\upsilon}\frac{c^2l^6h^2}{576(1+\upsilon)^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{i=1}^{k}C_{2n}E_i\alpha_i\langle-\{\exp[-\alpha_{2n,m}^2 t]-1\}\\
&\times\Big\{\psi''_{i,n,m}(\xi)ce_{2n}(\eta)+\upsilon\psi_{i,n,m}(\xi)ce''_{2n}(\eta)-\frac{h^2c^2(1-\upsilon)}{2}\\
&\times\big[\sinh 2\xi(\psi'_{i,n,m}(\xi)ce_{2n}(\eta))-\sin 2\eta(\psi_{i,n,m}(\xi)ce'_{2n}(\eta))\big]+\frac{576(1+\upsilon)^2}{c^2l^6h^2}\exp[-\alpha_{2n,m}^2 t]\psi_{i,n,m}(\xi)ce_{2n}(\eta)\rangle\\
&\times\Bigg\{\sum_{p=1}^{\infty}\Bigg[(\cos\beta_p\ell+\beta_p\ell\sin\beta_p\ell-1)/\beta_p^2\Bigg]\Bigg\{\bar{\bar{\theta}}_0+\sqrt{\frac{2}{\ell}}\frac{\kappa_i}{\lambda_i}\cos(\beta_p z_0)\times\bar{Q}_i(q_{2n,m})\Bigg\}\\
&-\int_0^t\exp[\alpha_{2n,m}^2\tau]\bar{G}(q_{2n,m},\beta_p,\tau)d\tau\Bigg\} \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
M_{\eta}^{(i)} &= \sqrt{\frac{2}{\ell}}\frac{1}{1-\upsilon}\frac{c^2l^6h^2}{576(1+\upsilon)^2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{i=1}^{k}C_{2n}E_i\alpha_i\langle-\{\exp[-\alpha_{2n,m}^2 t]-1\}\\
&\times\Big\{\upsilon\psi''_{i,n,m}(\xi)ce_{2n}(\eta)+\psi_{i,n,m}(\xi)ce''_{2n}(\eta)+\frac{h^2c^2(1-\upsilon)}{2}\\
&\times\big[\sinh 2\xi(\psi'_{i,n,m}(\xi)ce_{2n}(\eta))-\sin 2\eta(\psi_{i,n,m}(\xi)ce'_{2n}(\eta))\big]+\frac{576(1+\upsilon)^2}{c^2l^6h^2}\exp[-\alpha_{2n,m}^2 t]\psi_{i,n,m}(\xi)ce_{2n}(\eta)\rangle\\
&\times\Bigg\{\sum_{p=1}^{\infty}\Bigg[(\cos\beta_p\ell+\beta_p\ell\sin\beta_p\ell-1)/\beta_p^2\Bigg]\Bigg\{\bar{\bar{\theta}}_0+\sqrt{\frac{2}{\ell}}\frac{\kappa_i}{\lambda_i}\cos(\beta_p z_0)\times\bar{Q}_i(q_{2n,m})\Bigg\}\\
&-\int_0^t\exp[\alpha_{2n,m}^2\tau]\bar{G}(q_{2n,m},\beta_p,\tau)d\tau\Bigg\} \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
M_{\xi\eta}^{(i)} &= -\frac{c^2l^6h^2}{576(1+\upsilon)^2}\sqrt{\frac{2}{\ell}}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{i=1}^{k}\{\exp[-\alpha_{2n,m}^2 t]-1\}\alpha_i C_{2n}\\
&\times\Bigg[\sin 2\eta\psi'_{i,n,m}(\xi)ce_{2n}(\eta)+\sinh 2\xi\psi_{i,n,m}(\xi)ce'_{2n}(\eta)-\frac{2}{h^2c^2}\psi'_{i,n,m}(\xi)ce'_{2n}(\eta)\Bigg]\Bigg\{\sum_{p=1}^{\infty}\Bigg[(\cos\beta_p\ell\\
&+\beta_p\ell\sin\beta_p\ell-1)/\beta_p^2\Bigg]\times\Bigg[\bar{\bar{\theta}}_0+\sqrt{\frac{2}{\ell}}\frac{\kappa_i}{\lambda_i}\delta(t)\cos(\beta_p z_0)(\bar{Q}_i(q_{2n,m})\Bigg]\\
&-\int_0^t\exp[\alpha_{2n,m}^2\tau]\bar{G}(q_{2n,m},\beta_p,\tau)d\tau\Bigg\}/q_{2n,m} \qquad (34)
\end{aligned}
$$

The resulting equations of stresses can be obtained by substituting the equations (32)-(34) in equations (19)-(21). The equations of stresses are rather lengthy. Subsequently, the stress equations were omitted here for the sake of brevity, but were considered during graphical discussion utilizing MATHEMATICA software.

Numerical Results, Discussion and Remarks

We introduce the following dimensionless values

$$
\left.\begin{aligned}
&\bar{\xi}_i=\xi_i/b,\ \bar{a}=a/b,\ \bar{b}=b/b, \bar{z}=z/b,\ \bar{\ell}=\ell/b,\\
&e=c/b,\ \tau=\kappa t/b^2,\ \bar{\theta}=\theta/\theta_0,\ \bar{w}_i=w_i/\alpha_t\theta_0\ell,\\
&\bar{M}_{ij}=\sigma_{ij}/E\ell^3\quad(i,j=\xi,\eta),\ \bar{\sigma}_{ij}=\sigma_{ij}/E\alpha_t\theta_0,
\end{aligned}\right\} \qquad (35)
$$

For the sake of simplicity of numerical calculations, a two-layered elliptic annulus composite plate was considered. The numerical computations were carried out for Aluminium and Tin metal whose initial temperature was $0°C$ and for $(t > 0)$ the temperature raised to limited value. The physical parameters are considered as $\xi_1 = a = 0.2$cm, $\xi_3 = b = 0.9$cm, $\xi_2 = 0.35$cm, $\ell = 0.08$cm, $f_1(z,t) = 100$cm-$°C$, $f_2(z,t) = 20$cm-$°C$.

The mechanical material properties are considered in Table 1. as

Properties	Aluminium	Tin
Specific heat	$C_{v1} = 0.181$cal/g°C	$C_{v2} = 0.181$cal/g°C
Modulus of elasticities	$E_1 = 6.9 \times 10^6$N/cm^2	$E_2 = 4.7 \times 10^6$N/cm^2
Shear modulus	$G_1 = 2.7 \times 10^6$N/cm^2	$G_2 = 1.8 \times 10^6$N/cm^2
Poisson ratio	$\upsilon_1 = 0.35$	$\upsilon_2 = 0.36$
Thermal expansion coefficient	$\alpha_1 = 24.8 \times 10^{-6}$cm/cm-°C	$\alpha_2 = 23.0 \times 10^{-6}$cm/cm-°C
Thermal conductivity	$\lambda_1 = 0.52$cal sec^{-1}/cm-°C	$\lambda_2 = 0.15$cal sec^{-1}/cm-°C

The transcendental equation for determining the eigenvalues is given as

$$\begin{vmatrix} \alpha_1\phi'(\xi_1) - h_0\phi(\xi_1) & 0 & \alpha_1\varphi'(\xi_1) - B_{1n}\varphi(\xi_1) & 0 \\ 0 & \alpha_2\phi'(\xi_3) + h_2\phi(\xi_3) & 0 & \alpha_2\varphi'(\xi_3) + h_2\varphi(\xi_3) \\ \alpha_1\phi'(\xi_3) & \alpha_1\varphi'(\xi_3) & -\alpha_2\phi'(\xi_2) & -\alpha^2\varphi'(\xi_2) \\ 0 & \phi'(\xi_2)/R_1 & 0 & \varphi'(\xi_2)/R_1 \end{vmatrix} = 0 \qquad (36)$$

Substituting the dimensionless value of equation (35) in equations (28), (31) and in its stress components, the expressions for the temperature, deflection, and stresses were obtained respectively for the numerical discussion. In order to examine the influence of heating on the plate, a numerical calculation was performed for all variables, and numerical calculations are depicted in the following figures with the help of MATHEMATICA software. Figs. 2 to 4 illustrate the numerical results of temperature distribution, thermal deflection, stresses and bending moments of the elliptical plate due to interior heat generation within the solid. As shown in Fig. 2a, the temperature approaches to a minimum value at both extreme ends i.e. at $\eta = 0$ and $\eta = \pi$ due to more compressive force, whereas due to a tensile force, the temperature is high at centre i.e. at $\eta = \pi/2$, which gives an overall bell-shaped curve for both layers of different materials. Temperature trend in Fig. 2b increases gradually towards the outer end of each layer due to the combined effect of sectional and internal heat. Fig. 2c shows the slight increase in temperature along z-direction due to available internal heat source and heat accumulated due to sectional heat supply. Fig. 3a shows two layers deflection for different materials having centrally symmetric nature with maximum magnitude at $\eta = \pi/2$; then it approaches to zero at both extreme ends. As expected in Fig. 3b, the increased trend of deflection in each layer was observed. In Fig. 4a, the radial stresses and angular stresses are following the normal curve in nature for different material. Here, it is observed that they are zero at both extreme ends and giving maximum magnitude at the centre at $\eta = \pi/2$, whereas shear stresses are exactly sinusoidal in trend. Fig. 4b shows characteristic nature of radial stress, angular stress, and shear stress of the same pattern for different magnitude along the radial direction, but Fig. 4c shows linear nature of all stresses along the z-direction. Fig. 5a indicates that the bending moments, $M_\xi^{(i)}$ and $M_\eta^{(i)}$, along η-direction are maximum at both ends, whereas, they show the low trend at the centre giving cup-shaped symmetry. The moment $M_{\xi\eta}^{(i)}$ along angular direction shows sinusoidal nature. Fig. 5b depicts both $M_\xi^{(i)}$ and $M_\eta^{(i)}$ as stable to certain values then it increases very slowly but $M_{\xi\eta}^{(i)}$ shows a decreasing trend along ξ-direction.

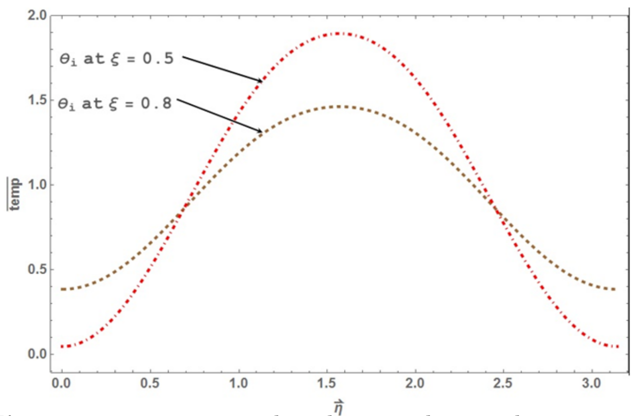

Fig. 2a. Temperature distribution along η-direction.

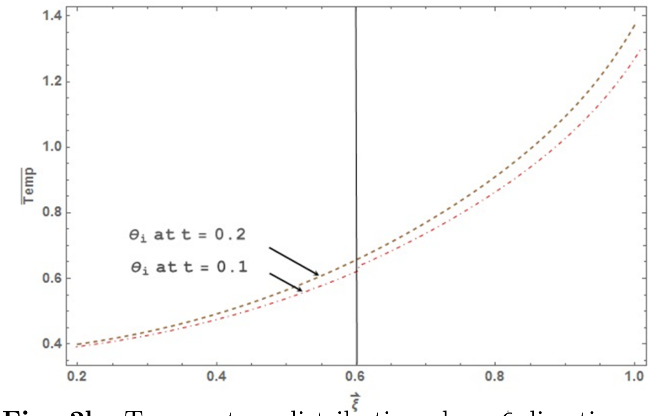

Fig. 2b. Temperature distribution along ξ-direction.

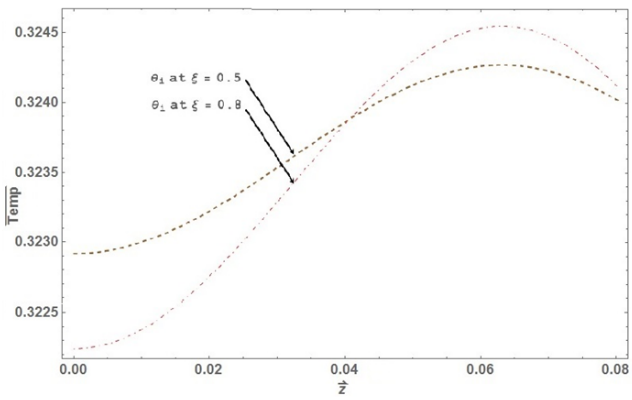

Fig. 2c. Temperature distribution along z-direction.

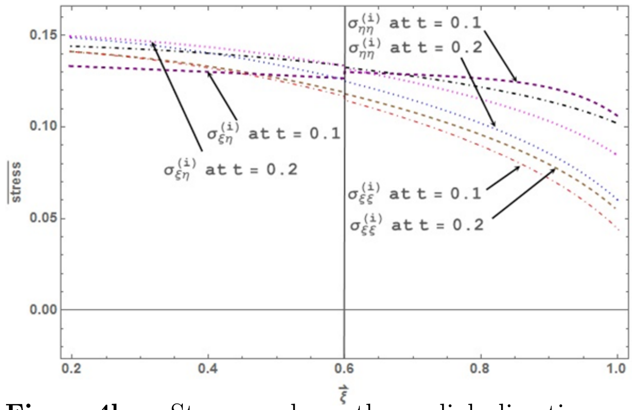

Fig. 4b. Stresses along the radial direction.

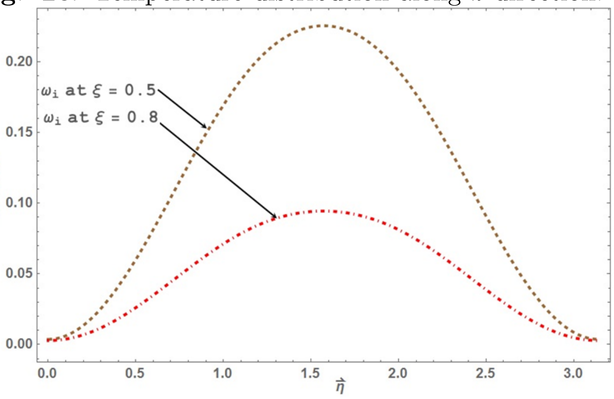

Fig. 3a. $w_i(\xi, \eta, t)$ along angular direction.

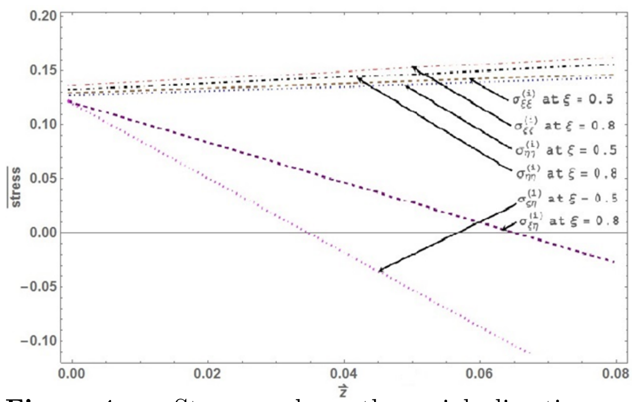

Fig. 4c. Stresses along the axial direction.

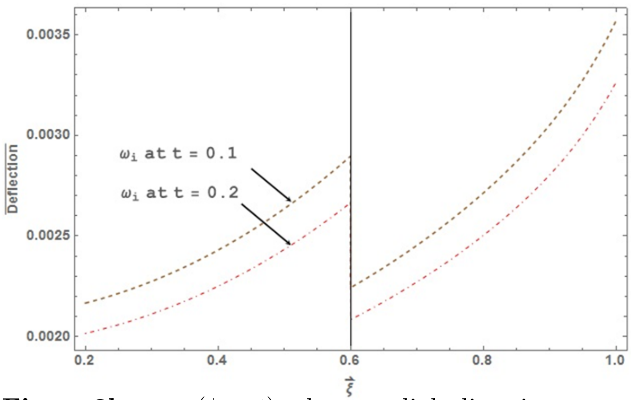

Fig. 3b. $w_i(\xi, \eta, t)$ along radial direction.

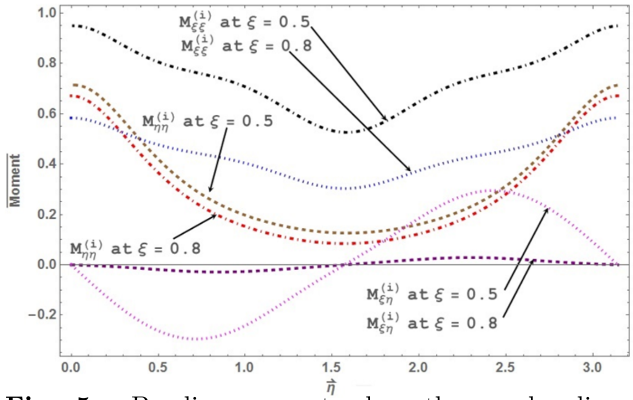

Fig. 5a. Bending moments along the angular direction.

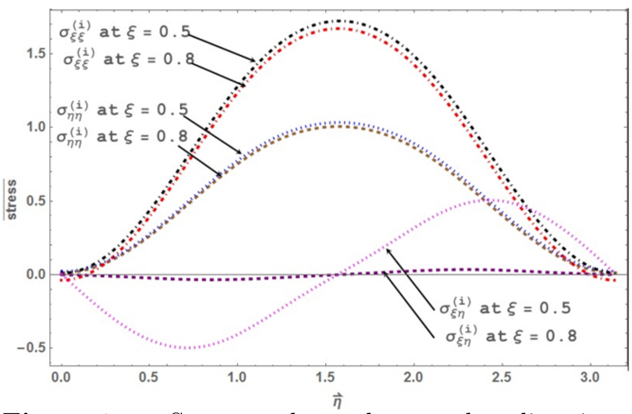

Fig. 4a. Stresses along the angular direction

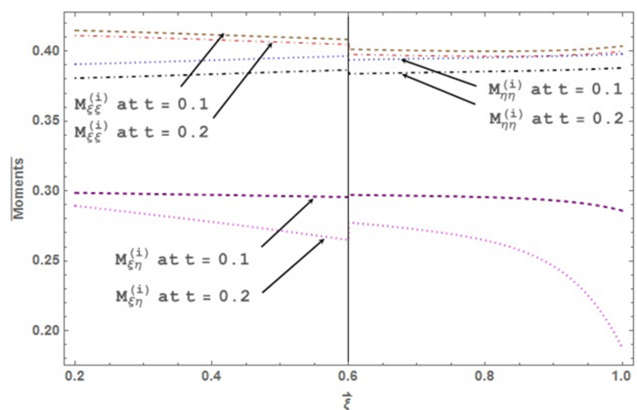

Fig. 5b. Bending moments along the radial direction.

Fig. 6a illustrates the dimensionless temperature distribution along time variation for different radial directions of the plate. The maximum value of temperature magnitude occurs at the outer edge due to additional heat supply with available internal heat energy throughout the body. The distribution of the dimensionless temperature gradient at each inner radii decreases in the low heated area of the central part of ellipse boundary tending below in one direction.

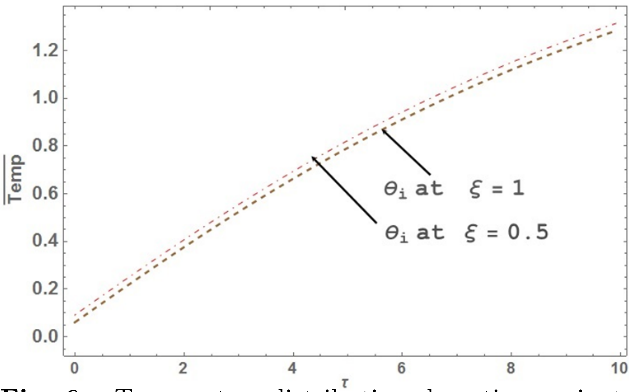

Fig. 6a. Temperature distribution along time variant.

Transition to Composite Circular Plate

When the elliptic composite plate tends to a composite circular plate of radius $r_i \leq r \leq r_{i+1}$, the semi-focal $c \to 0$ and therefore λ_m is the root of the transcendental equation $\psi_{i,n}(\alpha_m) = A_{in}\phi(\alpha_m) + B_{in}\varphi(\alpha_m) = 0$. Also $e \to 0$ as $\xi \to \infty$, $\sinh\xi \to \cosh\xi$, $h\cosh\xi \to r$ [as $h \to 0$], $\cosh\xi d\xi \to rdr$, $\cosh 2\xi d\xi \to 2\cosh 2\xi \sinh 2\xi d\xi \to 2rdr/h^2$, $h\sinh\xi d\xi \to dr$.
Using results [19]

$$\psi_{i,0}(\xi, q_{0,m}) \to p'_0\phi_{i,0}(\lambda_m r), \psi'_{i,0}(\xi, q_{0,m}) \to p'_0\phi'_{i,0}(\lambda_m r),$$

$$\psi''_{i,0}(\xi, q_{0,m}) \to p'_0\phi''_{i,0}(\lambda_m r), ce_0(\eta, q_m) \to 1/\sqrt{2},$$

$$A_0^{(0)} \to 1/\sqrt{2}, A_2^{(0)} \to 0, \Theta_{2m} \to 0,$$

$$\lambda_{0,m}^2 = \alpha_{0,m}^2/a^2 = \alpha_m^2/a^2 = \lambda_m^2,$$

$$p'_0 = Ce_0(0, q_{0,m})ce_0(2\pi, q_{0,m})/A_0^{(0)}.$$

Then, equation (36) degenerates into

$$\theta_i(r,z,t) = \sqrt{\frac{1}{\ell}} \sum_{m=1}^{\infty} C_m\phi_{i,m}(r) \sum_{i=1}^{k} \exp[-\lambda_m^2 t] \times \sum_{p=1}^{\infty} \cos(\beta_p z)\left[\theta_0 + \sqrt{\frac{2}{\ell}}\frac{\kappa_i}{\lambda_i}\cos(\beta_p z_0)\bar{Q}_i(\lambda_m)\right] - \int_0^t \exp[\lambda_m^2\tau]\bar{G}(\lambda_m, \beta_p, \tau)d\tau \quad (37)$$

in which the normalising constant can be given as

$$C_m \int_{r_i}^{r_{i+1}} (2r/h^2)\Phi_{i,n}(r)\Phi_{i,s}(r)dr = \begin{cases} 0, & \text{if } n \neq s \\ 1, & \text{if } n = s \end{cases} \quad (38)$$

The results above are in good agreemenr with the results [2].

2. Conclusions

The proposed analytical solution of transient thermal stress problem in multilayer elliptical composite regions was dealt in an elliptical coordinates system with the presence of a source of internal heat. To the authors cognizance, there have been no reports of the solution so far in which sources are generated according to the linear function of time in the mediums in the form of an elliptical plate of finite height with the interfaces with imperfect thermal contact under arbitrary initial temperature distribution. The analysis of non-stationary three-dimensional equation of heat conduction was investigated with the integral transformation method by establishing Sturm-Liouville integral transform considering series expansion function in terms of Eigenfunction of Sturm-Liouville boundary value problem. The following results were obtained through the research

- The advantage of this method is its generality and its mathematical power to handle various types of mechanical and thermal boundary conditions.

- The maximum tensile stress shifting from central core to outer region may be due to heat, stress, concentration or available internal heat sources under considered temperature field.

- The maximum tensile stress occurring in the circular core on the major axis compared to elliptical central part designates the distribution of impotent heating. It might be due to deficient perforation of heat through the elliptical inner surface.

- Finally, it was recognized that the temperature variation and thermal stresses of the two-layered composite elliptic annulus plate, using both ANSYS Workbench (refer Appendix B) and our outcomes, were both virtually equipollent.

Appendix A:

The Transformation and Its Essential Property

Consider a system of equations for composite region consisting of k-layers by the system

$$\alpha_i L[\psi_{i,n,m}(\xi)ce_{2n}(\eta)] = 2\beta_i q_{2n,m}(\cosh 2\xi - \cos 2\eta)\psi_{i,n,m}(\xi)ce_{2n}(\eta)] \tag{A1}$$

$$q_{2n,m} \geq 0 \;\; \xi_i \leq \xi \leq \xi_{i+1},\;\; i = 1, 2, 3\cdots k,\;\; 0 \leq \eta \leq 2\pi$$

subjected to the boundary conditions and interfacial boundary conditions

$$\left.\begin{aligned}
&-\alpha_1\psi_{i,n,m}(\xi),\xi\Big|_{\xi=\xi_1} = -h_0\psi_{i,n,m}(\xi)\Big|_{\xi=\xi_1},\;\; h_0 \geq 0\\
&-\alpha_k\psi_{k,n,m}(\xi),\xi\Big|_{\xi=\xi_{k+1}} = -h_k\psi_{k,n,m}(\xi)\Big|_{\xi=\xi_{k+1}},\;\; h_k \geq 0\\
&\alpha_i\psi_{i,n,m}(\xi),\xi\Big|_{\xi=\xi_{i+1}} = \alpha_{i+1}\psi_{i+1,n,m}(\xi),\xi\Big|_{\xi=\xi_{i+1}}\\
&[\psi_{i+1,n,m}(\xi) - \psi_{i,n,m}(\xi)]\Big/ R_i\Big|_{\xi=\xi_{i+1}},\\
&i = 1, 2, \cdots, (k-1)
\end{aligned}\right\} \tag{A2}$$

where $L = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$, Eigenfunction of the i^{th} layer is represented by $\psi_{i,n}(\xi)$, $q_{2n,m}$ is the Eigenvalue of the problem; α_i, β_i stand for the characteristics of the i^{th} layer, R_i denotes the characterises of the i^{th} interface, h_0 for the surface coefficients at $\xi = \xi_1$, and h_k for the surface coefficients at $\xi = \xi_{k+1}$, respectively. The general solution of equation (A1) is of the form

$$\Phi_{i,n,m}(\xi,\eta) = \psi_{i,n,m}(\xi)ce_{2n}(\eta) = [A_{in}\phi(\xi) + B_{in}\varphi(\xi)]ce_{2n}(\eta) \tag{A3}$$

satisfying the following boundary conditions, we get a system of $2k$ simultaneous equations so that arbitrary constants A_{in} and B_{in} can be obtained. Also from this $2k$ system of equations, we get the frequency equation by eliminating A_{in} and B_{in}. After substituting the values of A_{in} and B_{in}, the required solution of the Sturm-Liouville problem (A1) subjected to the boundary and interfacial conditions will be achieved (A2).

Orthogonality of the Eigenfunction $\psi_{i,n,m}$

If $\Phi_{i,n,m}$ and $\Phi_{i,s,r}$ be the solutions of equation (A1), then we have

$$\alpha_i L\Phi_{i,n,m}(\xi,\eta) = 2\beta_i q_{2n,m}(\cosh 2\xi - \cos 2\eta)\Phi_{i,n,m}(\xi,\eta) \tag{A4}$$

$$\alpha_i L\Phi_{i,s,r}(\xi,\eta) = 2\beta_i q_{2s,r}(\cosh 2\xi - \cos 2\eta)\Phi_{i,s,r}(\xi,\eta) \tag{A5}$$

Multiplying (A4) by $\Phi_{i,s,r}(\xi,\eta)$ and (A5) by $\Phi_{i,n,m}(\xi,\eta)$ and then substracting one leads to

$$\begin{aligned}
&\alpha_i[\Phi_{i,s,r}(\xi,\eta)\Phi_{i,n,m}(\xi,\eta),_\xi - \Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta),_\xi],_\xi\\
&+ [(\Phi_{i,s,r}(\xi,\eta)\Phi_{i,n,m}(\xi,\eta),_\eta - \Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta),_\eta],_\eta\\
&= 2\beta_i(q_{2n,m} - q_{2s,r})(\cosh 2\xi - \cos 2\eta)\Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta)
\end{aligned} \tag{A6}$$

Integrating with respect to η within 0 to 2π and with respect to ξ within ξ_i to ξ_{i+1}, $i = 1, 2, 3, \cdots, k$

$$\begin{aligned}
&\alpha_i \int_0^{2\pi} [\Phi_{i,s,r}(\xi,\eta)\Phi_{i,n,m}(\xi,\eta),_\xi - \Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta),_\xi]_{\xi_i}^{\xi_{i+1}} d\eta\\
&+ \int_{\xi_i}^{\xi_{i+1}} [\Phi_{i,s,r}(\xi,\eta)\Phi_{i,n,m}(\xi,\eta),_\eta - \Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta),_\eta]_0^{2\pi} d\xi\\
&= 2\beta_i(q_{2n,m} - q_{2s,r}) \int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}} (\cosh 2\xi - \cos 2\eta)\Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta)d\xi d\eta
\end{aligned} \tag{A7}$$

Due to the periodicity of the function, the second term on the right-hand side of equation (A7) vanishes. Thus, equation (A7) decreases to

$$\begin{aligned}
&\alpha_i \int_0^{2\pi} \sum_{i=1}^{k} [\Phi_{i,s,r}(\xi,\eta)\Phi_{i,n,m}(\xi,\eta),_\xi - \Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta),_\xi]_{\xi_i}^{\xi_{i+1}} d\eta\\
&= 2\sum_{i=1}^{k}\beta_i(q_{2n,m} - q_{2s,r}) \int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}} (\cosh 2\xi - \cos 2\eta)\Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta)d\xi d\eta
\end{aligned} \tag{A8}$$

Substituting equation (A3) into (A8), we obtain

$$
\begin{aligned}
&-2\pi A_0^{(2n)}\langle \alpha_k\psi_{k,s,r}(\xi_{k+1})[\psi_{k,n,m}(\xi),_{\xi}]_{\xi=\xi_{k+1}}\\
&-\alpha_k\psi_{k,n,m}(\xi_{k+1})\times[\psi_{k,s,r}(\xi),_{\xi}]_{\xi=\xi_{k+1}}\\
&+\sum_{i=1}^{k-1}\{\alpha_i\psi_{i,s,r}(\xi_{i+1})[\psi_{i,n,m}(\xi),_{\xi}]_{\xi=\xi_{i+1}}\\
&-\alpha_i\psi_{i,n,m}(\xi_{i+1})[\psi_{i,s,r}(\xi),_{\xi}]_{\xi=\xi_{i+1}}\\
&-[\alpha_{i+1}\psi_{i+1,s,r}(\xi_{i+1})[\psi_{i+1,n,m}(\xi),_{\xi}\\
&-\alpha_{i+1}\psi_{i+1,n,m}(\xi_{i+1})]_{\xi=\xi_{i+1}}[\psi_{i,s,r}(\xi),_{\xi}]_{\xi=\xi_{i+1}}]\\
&-[\alpha_1\psi_{1,s,r}(\xi_1)[\psi_{1,n,m}(\xi),_{\xi}]_{\xi=\xi_1}-\alpha_1\psi_{1,n,m}(\xi_1)\\
&[\psi_{1,s,r}(\xi),_{\xi}]_{\xi=\xi_{i+1}}]\}\rangle\\
&=2\beta_i(q_{2n,m}-q_{2s,r})\int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}}(\cosh 2\xi\\
&-\cos 2\eta)\Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta)d\xi d\eta \qquad \text{(A9)}
\end{aligned}
$$

Using the boundary and interfacial conditions (A2) for eigenfunctions, we yield

$$
\begin{aligned}
&-2\pi A_0^{(2n)}\langle\psi_{k,s,r}(\xi_{k+1})[-h_k\psi_{k,n,m}(\xi_{k+1})]\\
&-\alpha_k\psi_{k,n,m}(\xi_{k+1})\times[-h_k\psi_{k,s,r}(\xi_{k+1})]\\
&+\sum_{i=1}^{k-1}\{\psi_{i,s,r}(\xi_{i+1})[\psi_{i+1,n,m}(\xi_{i+1})\\
&-\psi_{i,n,m}(\xi_{i+1})/R_i-\psi_{i,n,m}(\xi_{i+1})[\psi_{i+1,s,r}(\xi_{i+1})\\
&-\psi_{i,s,r}(\xi_{i+1})]/R_i-\psi_{i+1,s,r}(\xi_{i+1})[\psi_{i+1,n,m}(\xi_{i+1})\\
&-\psi_{i,n,m}(\xi_{i+1})]/R_i+\psi_{i+1,n,m}(\xi_{i+1})\}[\psi_{i+1,s,r}(\xi_{i+1})\\
&-\psi_{i,s,r}(\xi_{i+1})]/R_i\}-[\psi_{1,s,r}(\xi_1)h_0\psi_{1,n,m}(\xi_1)\\
&-\psi_{1,n,m}(\xi_1)h_0\psi_{1,s,r}(\xi_1)]\rangle=2\beta_i(q_{2n,m}-q_{2s,r})\int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}}\\
&(\cosh 2\xi-\cos 2\eta)\Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta)d\xi d\eta \qquad \text{(A10)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&C_{2n}\int_0^{2\pi}\int_{\xi_i}^{\xi_{i+1}}(\cosh 2\xi\\
&-\cos 2\eta)\Phi_{i,n,m}(\xi,\eta)\Phi_{i,s,r}(\xi,\eta)d\xi d\eta\\
&=\begin{cases}0, & \text{if } n\neq s,\ m\neq r\\ 1, & \text{if } n=s,\ m=r\end{cases} \qquad \text{(A11)}
\end{aligned}
$$

in which C_{2n} is a normalizing constant.

Property of the Transform

Let us consider the effect of transform defined above

$$
\begin{aligned}
&\sum_{i=1}^{k}\int_0^{2\pi}\int_{\xi_i}^{\xi i+1}\beta_i(\cosh 2\xi-\cos 2\eta)[\psi_{i,n,m}(\xi)ce_{2n}(\eta)]\\
&\times\left(\frac{\alpha_i}{\beta_i(\cosh 2\xi-\cos 2\eta)}\right)[f_i(\xi,\eta),_{\xi\xi}+f_i(\xi,\eta),_{\eta\eta}]\\
&=-2\pi A_0^{(2n)}\langle\{\psi_{k,n,m}(\xi_{k+1})[\alpha_k f_k(\xi,\eta),_{\xi}-h_k f_k(\xi,\eta)]_{\xi}\\
&=\xi_{k+1}+\sum_{i=1}^{k-1}\{\psi_{i,n,m}(\xi_{i+1})[\alpha_i f_i(\xi,\eta),_{\xi}-[f_{i+1}(\xi,\eta)\\
&-f_i(\xi,\eta)]/R_i]_{\xi}=\xi_{i+1}-\psi_{i+1,n,m}(\xi_{i+1})[\alpha_{i+1}f_{i+1}(\xi,\eta),_{\xi}\\
&-[f_{i+1}(\xi,\eta)-f_i(\xi,\eta)]/R_i]_{\xi}=\xi_{i+1}-\psi_{1,n,m}(\xi_1)[\alpha_1 f_1(\xi,\eta),_{\xi}\\
&-h_0 f_1(\xi,\eta)]_{\xi=\xi_1}\rangle-2q_{2n,m}\sum_{i=1}^{k}\bar{f}_i(q_{2n,m}) \qquad \text{(A12)}
\end{aligned}
$$

Hence (A12) is the fundamental property of the Sturm-Liouville transform for the composite region.

Appendix B:

Thermoelastic Analysis Through ANSYS Software

The well-known software ANSYS that could simulate static problem was used to calculate temperature and stress distribution of a two-layered elliptic annulus composite plate under thermal load. First, temperature changes on the composite plate have were computed, and then the thermal stress as a result of temperature changes was carried out. Finally, the distribution of temperature and thermal stress versus time were plotted to validate our results.

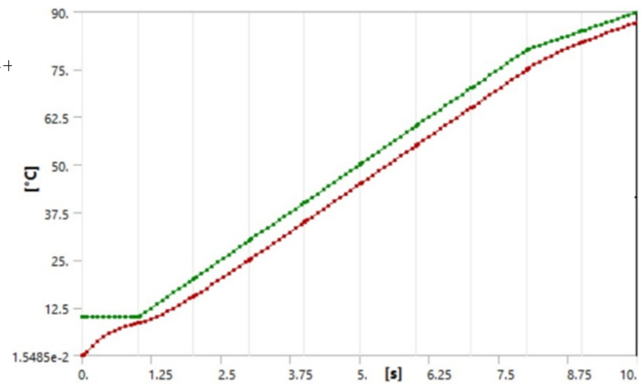

Fig. 7a. Thermal stresses along the time variant.

The Fig. 7a represents the temperature distribution of two layers varying with the time obtain using ANSYS which is very much analogous to the temperature distribution obtained in Fig. 6b for various fixed radius values. Table 1 provides detailed numerical values of the temperature distribution on a two-layered elliptic annulus composite plate with increasing time span.

Table 1
Temperature values for different time.

Time [s]	Minimum [°C]	Maximum [°C]
0.00	0.1814	10.0
1.24	9.5294	12.4
2.54	20.467	25.4
3.74	32.293	37.4
5.00	44.867	50.0
6.24	57.255	62.4
7.54	70.242	75.4
8.74	80.475	83.7
10.0	87.331	90.0

Fig. 7b represents the influence of the thermal stress field with time which is very much analogous to the thermal stress distribution with time obtained in Fig. 4b. The maximum tensile stress obtained on the centre core (i.e. inner radius) due to the additional sectional heat supply, whereas minimum temperature changes occur on the outer boundary of the elliptical plate.

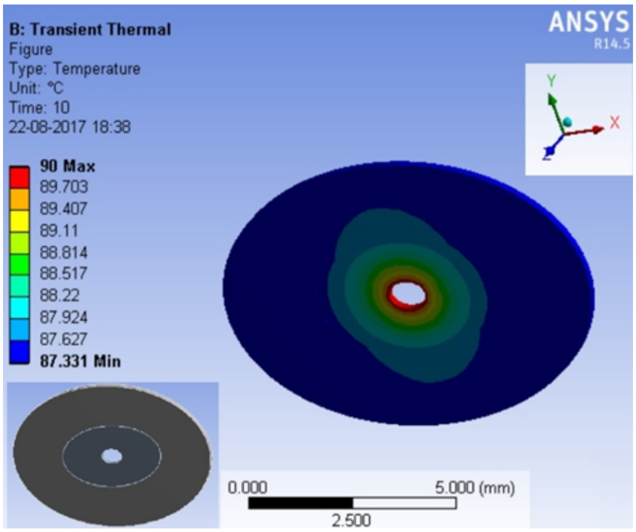

Fig. 7b. Thermal stresses along the time variant.

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